FINAL PERIOD OF DECAY OF TURBULENT MOTIONS OF VISCOELASTIC FLUIDS

V. A. Gorodtsov and V. P. Myasnikov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 97–100, 1965

The Newtonian model of a viscous fluid defined by two dimensional parameters* (density ρ and kinematic viscosity ν) gives a good description of the flow of structureless fluids.** From the thermodynamic viewpoint nonhomogeneous flow of a viscous fluid is linked with the process of approach to the equilibrium state, whereas the Newtonian model describes the state of spatially distributed nonequilibrium. For motions in the region of the scale l the characteristic time of approach to the equilibrium state $\tau = l^2\nu^{-1}$, i.e., is largely determined by this scale. Similar considerations also apply to such phenomena as diffusion and heat conduction.

However, in real fluids other physical processes also occur and these admit local description. If local relaxation processes (such as orientation of particles, reorganization of supramolecular structures, adaptation of the motion of an admixture to the motion of the fluid etc.) take place in the medium and exert a strong influence on its mechanical behavior, then given a phenomenological approach to the analysis of motion it is necessary to introduce at least a relaxation time θ characterizing the rate of approach to the equilibrium state of the most important of all the processes involved.

The simplest fluid, for both distributed and local nonequilibrium, will be the fluid defined by the three dimensional parameters ρ, ν, θ (viscoelastic fluid). Note that from these parameters it is possible to construct the characteristic length $\theta^{1/2}\nu^{1/2}$, velocity $\nu^{1/2}\theta^{-1/2}$, and modulus of elasticity $\rho\nu\theta^{-1}$. This further confirms that models ν and θ characterize the motion of structured fluids. The best known phenomenological model of this type is the Maxwell fluid with stress relaxation.

It is natural to expect a clear manifestation of the special features of the mechanical behavior of such fluids in turbulent flows, when motions with different space and time scales occur. Our research is concerned with the simplest stage of decay of turbulent motion of a viscoelastic fluid with constant ν and θ, when the higher correlation moments of the velocity and stress fields can be neglected as compared with the second moments. In this case we need not concern ourselves with the question of the possibility of a different choice of nonlinear terms with respect to velocities and stresses in specific models of viscoelastic fluids.

§1. **Decay in model with stress relaxation.** It is natural to describe the turbulent motions of such a fluid in terms of random velocity v_i and stress σ_{ij} fields. Below we shall restrict ourselves to a study of the decay of homogeneous isotropic turbulence. In the final period of decay the equations of state and the dynamic equations can be written directly in linearized form

$$\frac{\partial v_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad \frac{\partial v_i}{\partial x_i} = 0, \tag{1.1}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + \frac{1}{\theta}\sigma_{ij} = \frac{\nu}{\theta}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Here

$$\langle v_i \rangle = \langle \sigma_{ij} \rangle = 0, \qquad p = P - \langle P \rangle, \tag{1.2}$$

where P is the pressure in the fluid. The true values of the pressures and shear stresses are equal respectively to ρp and $\rho\sigma_{ij}$, so that p and σ_{ij} are kinematic quantities.

We introduce the following correlation tensors of the random fields in question:

$$\langle v_i(\mathbf{x})\, v_j(\mathbf{x}+\mathbf{r})\rangle = R_{ij}(\mathbf{r}), \qquad \langle v_i(\mathbf{x})\, \sigma_{jk}(\mathbf{x}+\mathbf{r})\rangle = S_{ijk}(\mathbf{r}),$$
$$\langle \sigma_{ij}(\mathbf{x})\, p(x+r)\rangle = T_{ij}(\mathbf{r}), \qquad \langle \sigma_{ij}(\mathbf{x})\, \sigma_{kl}(\mathbf{x}+\mathbf{r})\rangle = W_{ijkl}(\mathbf{r}). \tag{1.3}$$

From system (1.1) for these tensors we get the system

$$\frac{\partial R_{ij}(\mathbf{r})}{\partial t} = \frac{\partial}{\partial r_k}[S_{ijk}(\mathbf{r}) + S_{jik}(\mathbf{r})], \qquad \frac{\partial S_{ikl}(\mathbf{r})}{\partial t} + \frac{1}{\theta}S_{ikl}(\mathbf{r}) = \frac{\nu}{\theta}\left(\frac{\partial R_{ik}}{\partial r_l} + \frac{\partial R_{il}}{\partial r_k}\right) + \frac{\partial T_{kl}(\mathbf{r})}{\partial r_i} - \frac{\partial W_{ijkl}(\mathbf{r})}{\partial r_j},$$
$$\frac{\partial W_{ijkl}(\mathbf{r})}{\partial t} + \frac{2}{\theta}W_{ijkl}(\mathbf{r}) = -\frac{\nu}{\theta}\left[\frac{\partial S_{kij}(\mathbf{r})}{\partial r_l} + \frac{\partial S_{lij}(\mathbf{r})}{\partial r_k} + \frac{\partial S_{ikl}(\mathbf{r})}{\partial r_j} + \frac{\partial S_{jkl}(\mathbf{r})}{\partial r_i}\right], \tag{1.4}$$
$$\frac{\partial R_{ij}(\mathbf{r})}{\partial r_j} = \frac{\partial R_{ij}(\mathbf{r})}{\partial r_i} = 0, \qquad \frac{\partial S_{ikl}(\mathbf{r})}{\partial r_i} = 0.$$

From the second of Eqs. (1.4) and the solenoidality of R_{ij} and S_{ikl} we have

$$\Delta T_{kl}(\mathbf{r}) = \frac{\partial^2 W_{ijkl}(\mathbf{r})}{\partial r_i \partial r_j}, \tag{1.5}$$

where Δ is the Laplace operator with respect to the variables r_m (m = 1,2,3).

Using the same letters to denote the Fourier transforms of these tensors, we will distinguish them from the correlation tensors only in respect of the argument.

By virtue of isotropy and the solenoidality of the tensors R_{ij} and S_{ikl} we have [1]

$$R_{ij}(\varkappa) = (\varkappa_i\varkappa_j - \varkappa^2\delta_{ij})\, R(\varkappa),$$
$$S_{ijk}(\varkappa) = [\varkappa_j(\varkappa_i\varkappa_k - \varkappa^2\delta_{ik}) + \varkappa_k(\varkappa_i\varkappa_j - \varkappa^2\delta_{ij})]\, S(\varkappa). \tag{1.6}$$

From the tensor equations (1.4) we can obtain scalar equations for the functions R ($\varkappa$), S ($\varkappa$) and for the function

$$W(\varkappa) = \varkappa_j\varkappa_l(\varkappa_i\varkappa_k - \varkappa^2\delta_{ik})\, W_{ijkl}(\varkappa). \tag{1.7}$$

These equations have the form

$$\frac{\partial R(\varkappa)}{\partial t} = 2i\varkappa^2 S(\varkappa),$$
$$\left(\frac{\partial}{\partial t} + \frac{1}{\theta}\right) S(\varkappa) = i\,\frac{\nu}{\theta} R(\varkappa) - \frac{i}{2\varkappa^6} W(\varkappa), \tag{1.8}$$
$$\left(\frac{\partial}{\partial t} + \frac{2}{\theta}\right) W(\varkappa) = -\frac{4i\nu\varkappa^8}{\theta} S(\varkappa).$$

*Henceforth we shall consider only the simple case of an incompressible fluid.

**Under ordinary conditions short-range order is not manifested in low-molecular fluids.

System (1.8) reduces to a single equation for the determination of $R(\varkappa)$

$$\left(\frac{\partial}{\partial t}+\frac{1}{\theta}\right)\left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\frac{2}{\theta}\right)+\frac{4\nu\varkappa^2}{\theta}\right]R(\varkappa, t)=0\,. \quad (1.9)$$

For $t \gg 0$ and $\varkappa \to 0$ relation (1.9) reduces into the equation

$$\frac{\partial R(0,t)}{\partial t}=0\,, \quad (1.10)$$

which denotes the existence of a Loitsyanskii invariant in the final period of decay [1].

The solution of (1.9) has the following form:

$$R(\varkappa, t)=C_1(\varkappa, t_0)\exp\left(-\frac{t-t_0}{\theta}\right)+$$
$$+\exp\left(-\frac{t-t_0}{\theta}\right)\left[C_2(\varkappa, t_0)\exp\left(-\frac{t-t_0}{\theta}\sqrt{1-4\nu\theta\varkappa^2}\right)+\right.$$
$$\left.+C_3(\varkappa, t_0)\exp\left(\frac{t-t_0}{\theta}\sqrt{1-4\nu\theta\varkappa^2}\right)\right], \quad (1.11)$$

where t_0 is some initial moment of time at which it is already possible to neglect higher-order correlations. The functions C_1, C_2, and C_3 can be expressed in terms of the functions $R(\varkappa, t_0)$, $S(\varkappa, t_0)$ and $W(\varkappa, t_0)$.

Since in the final period of decay interactions between motions with different $\varkappa$ are neglected (nonlinear terms disregarded), they damp independently of each other.

Bearing in mind that there is a characteristic wave number

$$\varkappa_0={}^1/_2\nu^{-1/2}\theta^{-1/2} \quad (1.12)$$

dividing the wave space into two regions, we will consider the law of attenuation in the two regions separately.

At $\varkappa > \varkappa_\theta$, which corresponds to small-scale motions, we get

$$R(\varkappa, t)=[C_1(\varkappa, t_0)+C_2(\varkappa, t_0)e^{-i\omega(\varkappa)(t-t_0)}+$$
$$+C_3(\varkappa, t_0)e^{i\omega(\varkappa)(t-t_0)}]\exp\left(-\frac{t-t_0}{\theta}\right), \quad (1.13)$$

$$\omega(\varkappa)=\theta^{-1}\varkappa_\theta^{-1}(\varkappa^2-\varkappa_\theta^2)^{1/2}\,. \quad (1.14)$$

Equation (1.13) shows that the small-scale motions decay in accordance with the universal law $\exp[-(t-t_0)/\theta]$, while the oscillating behavior of the expression in square brackets reflects the fact that in the Maxwell model high-frequency elastic shear waves exist.

Conversely, at $\varkappa < \varkappa_\theta$, which corresponds to large-scale motions, for $t-t_0 \gg 0$

$$R(\varkappa, t)=C_3e^{-2\nu\varkappa^2(1+\nu\theta\varkappa^2+\ldots)(t-t_0)}. \quad (1.15)$$

By virtue of the solenoidality conditions, as $\varkappa \to 0$ the value of $C_3(\varkappa, t_0)$ will be bounded and equal to $C_3(0, t_0)$.

At sufficiently large $t-t_0$ the motions with high wave numbers decay and there remain only the large-scale motions with a law of attenuation analogous to that for an ordinary viscous fluid. In this case the correlation tensor decreases with time in accordance with the known law of $t^{-5/2}$ [1].

§2. Decay in a model with relaxation of shear strain rates. As another example of a medium with parameters ν, ϑ let us consider a model with shear strain rate relaxation. In the final period of decay the system of linearized equations can be written in the form

$$\frac{\partial v_i}{\partial t}=-\frac{\partial p}{\partial x_i}+\frac{\partial \sigma_{ij}}{\partial x_j}, \qquad \frac{\partial v_i}{\partial x_i}=0,$$
$$\sigma_{ij}=\nu\left(1+2\theta\frac{\partial}{\partial t}\right)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right). \quad (2.1)$$

A model of this type, taking into account local relaxation changes of the velocity field, has a quite clear physical meaning. Thus, in the macroscopic description of the motion of a fluid with suspended particles as a homogeneous medium, apart from the concentration change in the viscosity coefficient, it is necessary to take into account the adaptation of the motion of the particles to the motion of the fluid. The characteristic relaxation time of this process $\theta \sim (\rho_n/\rho)a^2\nu^{-1}$, where ρ_n, ρ are the density of the solid particles and the fluid, respectively, and a is the characteristic linear dimension of the particles.

Assuming isotropy and homogeneity, we can write the equation for $R_{ij}(\mathbf{r})$ as follows:

$$\frac{\partial}{\partial t}[(1-2\nu\theta\Delta)R_{ij}(\mathbf{r})]=2\nu\Delta R_{ij}(\mathbf{r}),$$
$$\frac{\partial}{\partial r_i}R_{ij}(\mathbf{r})=\frac{\partial}{\partial r_j}R_{ij}(\mathbf{r})=0\,. \quad (2.2)$$

For the Fourier transform $R_{ij}(\varkappa)$ from (2.2) we obtain the expression

$$R_{ij}(\varkappa, t)=(\varkappa_i\varkappa_j-\varkappa^2\delta_{ij})C(\varkappa, t_0)e^{-2\nu_\theta\varkappa^2(t-t_0)},$$
$$\nu_\theta=\frac{\nu}{1+2\nu\theta\varkappa^2}\,. \quad (2.3)$$

From (2.3) it is clear that with decay of turbulence the behavior of such a fluid is analogous to the law of decay for a viscous fluid, but with an effective kinematic viscosity coefficient depending on the wave number.

For $\varkappa \ll \varkappa_\theta$ (large-scale fluctuations)

$$R_{ij}(\varkappa, t)=(\varkappa_i\varkappa_j-\varkappa^2\delta_{ij})\,C(0, t_0)e^{-2\nu\varkappa^2(t-t_0)}, \quad (2.4)$$

where $C(0, t_0)$ is a bounded constant.

Conversely, for $\varkappa \gg \varkappa_\theta$ (small-scale fluctuations) we get the universal law of attenuation with respect to time $e^{-t/\theta}$.

The qualitatively obtained results for a model with shear strain rate relaxation coincide with the results obtained in the previous section.

This conclusion is not unexpected and reflects the general fact of the presence of two relaxation times. Indeed, we shall consider the behavior of fluctuations of scale l or correspondingly of the wave vector $\varkappa \sim 1/l$. If we consider the scales for which

$$\tau=\frac{l^2}{\nu}\sim\frac{1}{\varkappa^2\nu}\gg\theta,$$

then for such a scale at $t-t_0 \gg 0$ local relaxation processes will not play a part and the motion will be entirely determined by ordinary viscous relaxation.

The corresponding law of attenuation will be

$$R \sim e^{-2\nu\varkappa^2(t-t_0)}. \qquad (2.5)$$

Conversely, for fluctuations of scale l satisfying the condition

$$\tau \sim \frac{1}{\varkappa^2\nu} \ll \theta$$

and corresponding to the case where spatially distributed nonequilibrium rapidly relaxes, a decisive role at large $t - t_0$ is played only by local relaxation processes and the law of attenuation is

$$R \sim e^{-t/\theta}. \qquad (2.6)$$

According to [2], in the fluid model in question velocity jumps also damp exponentially (2.6). This is in agreement with the above, since when the jump is damped it is mainly the high frequencies of the spectrum that are distorted.

We note that a real medium is characterized by a whole spectrum of relaxation times; however, in the final period of decay the main role will be played by a certain minimal relaxation time.

The authors thank G. I. Barenblatt for his useful advice.

REFERENCES

1. G. K. Batchelor, The Theory of Homogeneous Turbulence [Russian translation], IL, 1955.

2. G. I. Barenblatt and G. G. Chernyi, "Moment relations at surfaces of discontinuity in dissipative media," PMM, vol. 27, no. 5, 1963.

20 May 1965 Moscow